

LAST TIME: Cross Product

9/3/21

ex. $\vec{u} = \langle 5, 3, -1 \rangle$, $\vec{v} = \langle 8, 4, 2 \rangle$ *show all steps!

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & -1 \\ 8 & 4 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & -1 \\ 4 & 2 \end{vmatrix} \vec{i} - \begin{vmatrix} 5 & -1 \\ 8 & 2 \end{vmatrix} \vec{j} + \begin{vmatrix} 5 & 3 \\ 8 & 4 \end{vmatrix} \vec{k}$$

$$= (3 \cdot 2 - (-1) \cdot 4) \vec{i} - (5 \cdot 2 - (-1) \cdot 8) \vec{j} + (5 \cdot 4 - 3 \cdot 8) \vec{k}$$

$$= (6 + 4) \vec{i} - (10 + 8) \vec{j} + (20 - 24) \vec{k} = 10\vec{i} - 18\vec{j} - 4\vec{k} = \boxed{\langle 10, -18, -4 \rangle}$$

↳ Recall: Prop (Algebraic Properties of Cross Product)

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$ and $c \in \mathbb{R}$

① $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

② $(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v})$ and $\vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$

③ $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

④ $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$

⑤ $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

⑥ $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})(\vec{v}) - (\vec{u} \cdot \vec{v})(\vec{w})$

⑦ \vec{u} and \vec{v} are both orthogonal to $\vec{u} \times \vec{v}$

⑧ $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$ with θ the angle between \vec{u}, \vec{v} } geometric properties

⑨ $\vec{u} \times \vec{v} = 0$ iff \vec{u} and \vec{v} are parallel.

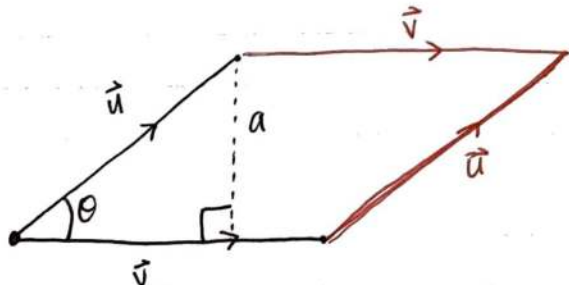
ex. Take $\vec{v} \times \vec{u}$ for the given vectors.

$$\vec{v} \times \vec{u} = -\vec{u} \times \vec{v}$$

$$= -\langle 10, -18, -4 \rangle$$

$$= \langle -10, 18, 4 \rangle$$

↳ $\vec{u} \times \vec{v}$ is computed w "right hand rule"



$$\sin(\theta) = \frac{a}{|\vec{v}|}$$

$$\therefore a = |\vec{v}| \sin(\theta)$$

$$\therefore \text{area of parallelogram}$$

determined by \vec{u} & \vec{v} is

$$A = (\text{base})(\text{height}) = |\vec{u}| a = |\vec{u}| |\vec{v}| \sin \theta$$

Proof of prop 8: We use the algebraic properties to compute $|\vec{u} \times \vec{v}|^2$

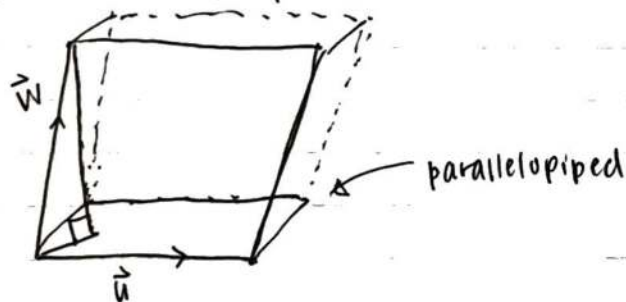
$$\begin{aligned}
 &= |\vec{u} \times \vec{v}|^2 = (\vec{u} \times \vec{v}) \cdot (\vec{u} \times \vec{v}) \quad (\text{Properties of dot product}) \\
 &= \vec{u} \cdot (\vec{v} \times (\vec{u} \times \vec{v})) \quad (\text{Part 5 of properties of cross}) \\
 &= \vec{u} \cdot ((\vec{v} \cdot \vec{v})\vec{u} - (\vec{v} \cdot \vec{u})\vec{v}) \quad (\text{Part 6 of properties of cross}) \\
 &= \vec{u} \cdot (\vec{v} \cdot \vec{v})\vec{u} - \vec{u} \cdot ((\vec{v} \cdot \vec{u})\vec{v}) \quad (\text{Properties of dot}) \\
 &= (\vec{v} \cdot \vec{v})(\vec{u} \cdot \vec{u}) - (\vec{v} \cdot \vec{u})(\vec{u} \cdot \vec{v}) \quad (\text{Properties of dot}) \\
 &= |\vec{v}|^2 |\vec{u}|^2 - (\vec{u} \cdot \vec{v})^2 \quad (\text{Properties of dot}) \\
 &= (|\vec{v}| |\vec{u}|)^2 - (|\vec{u}| |\vec{v}| \cos(\theta))^2 \quad \leftarrow \text{geometric interpretation of dot product} \\
 &= (|\vec{u}| |\vec{v}|)^2 - (|\vec{u}| |\vec{v}|)^2 \cos^2(\theta) \\
 &= (|\vec{u}| |\vec{v}|)^2 (1 - \cos^2(\theta)) \\
 &= (|\vec{u}| |\vec{v}|)^2 \sin^2(\theta) \\
 &= (|\vec{u}| |\vec{v}| \sin(\theta))^2 \\
 \therefore |\vec{u} \times \vec{v}|^2 &= (|\vec{u}| |\vec{v}| \sin(\theta))^2
 \end{aligned}$$

Recall: θ is the angle between vectors \vec{u} and \vec{v} , so $\theta \in [0, \pi]$, so $\sin(\theta) \geq 0$

Hence $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin(\theta)$

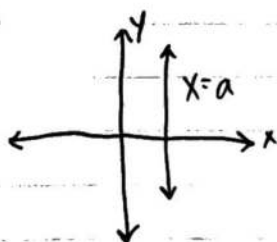
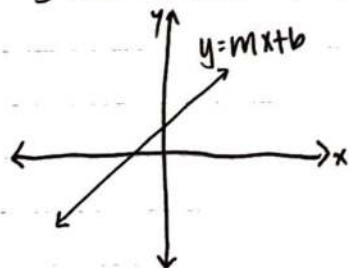
Cor: The magnitude $|\vec{u} \times \vec{v}|$ is the area of the parallelogram determined by \vec{u} & \vec{v}

prop: The scalar triple product $\vec{u} \cdot (\vec{v} \times \vec{w})$ is the signed volume of the parallelepiped by $\vec{u}, \vec{v}, \vec{w}$



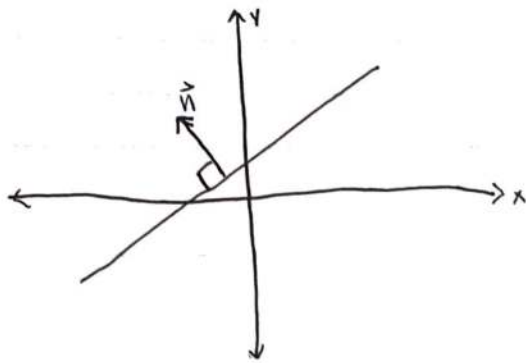
Proof is on website

§ 12.5: Lines & Planes



$$ax + by - c = 0$$

In 2-space:



line is the set of points w/ $\vec{n} \cdot \vec{x} = c$

- Generalize this equation to 3-space.

$$\vec{n} \cdot \vec{x} = d$$

i.e. $\langle a, b, c \rangle \cdot \langle x, y, z \rangle = d$

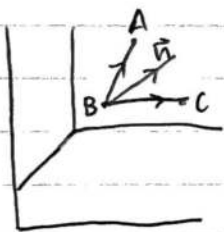
i.e. $ax + by + cz = d$

↳ This is a plane in 3-space

NB: If we know two nonparallel vectors \vec{u} & \vec{v} which lie in the plane (i.e. their head & tail can be expressed on the plane at the same time) then $\vec{n} = \vec{u} \times \vec{v}$ is a normal vector to the plane. i.e. it's perpendicular to every vector in the plane

ex: Find the vector equation of the plane through the points $(0, 1, 3)$, $(4, 9, 7)$, and $(1, 2, 3)$

solution:



Note that the vectors $\vec{u} = \langle 4-0, 9-1, 7-3 \rangle$
 $= \langle 4, 8, 4 \rangle$

and $\vec{v} = \langle 1-0, 2-1, 3-3 \rangle$
 $= \langle 1, 1, 0 \rangle$

lie in the desired plane

\therefore we may use normal vector $\vec{n} = \vec{u} \times \vec{v}$

$$\therefore \vec{n} = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 8 & 4 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= \langle -4, -(-4), -4 \rangle = \langle 1, -1, 1 \rangle$$

\therefore the plane has equation

$$\vec{n} \cdot \vec{x} = d \text{ for some constant } d$$

i.e. $\langle 1, -1, 1 \rangle \cdot \langle x, y, z \rangle = d$

i.e. $x - y + z = d$

↳ To compute d , use $(0, 1, 3)$

$$d = 0 - 1 + 3 = 2, \text{ so the plane has equation } x - y + z = 2$$